

# The thermal statistics of quasi-probabilities' analogs in phase space

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We focus attention upon the thermal statistics of the classical analogs of quasi-probabilities' (QP) in phase space for the important case of quadratic Hamiltonians. We consider the three more important OPs: 1) Wigner's,  $P$ -, and Husimi's. We show that, for all of them, the ensuing semiclassical entropy is a function *only* of the fluctuation product  $\Delta x \Delta p$ . We ascertain that *the semi-classical analog of the  $P$ -distribution* seems to become un-physical at very low temperatures. The behavior of several other information quantifiers reconfirms such an assertion in manifold ways. We also examine the behavior of the statistical complexity and of thermal quantities like the specific heat.

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## I. INTRODUCTION

A quasi-probability distribution is a mathematical construction that resembles a probability distribution but does not necessarily fulfill some of the Kolmogorov's axioms for probabilities [1]. Quasi-probabilities exhibit general features of ordinary probabilities. Most importantly, they yield expectation values with respect to the weights of the distribution. However, they disobey the third probability postulate [1], in the sense that regions integrated under them do not represent probabilities of mutually exclusive states. Some quasi-probability distributions exhibit zones of negative probability density. This kind of distributions often arise in the study of quantum mechanics when discussed in a phase space representation, of frequent use in quantum optics, time-frequency analysis, etc.

One usually considers a density operator  $\hat{\rho}$ , defined with respect to a complete orthonormal basis and shows that it can always be written in a diagonal manner, provided that an overcomplete basis is at hand [2]. This is the case of coherent states  $|\alpha\rangle$  [3], for which [2]

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha|. \quad (1)$$

We have  $d^2\alpha/\pi = dx dp/2\pi\hbar$ , with  $x$  and  $p$  phase space variables. Coherent states, right eigenstates of the annihilation operator  $\hat{a}$ , serve as the overcomplete basis in such a build-up [2, 3].

There exists a family of different representations, each connected to a different ordering of the creation and destruction operators  $\hat{a}$  and  $\hat{a}^\dagger$ . Historically, the first of these is the Wigner quasi-probability distribution  $W$  [4], related to symmetric operator ordering. In quantum optics the particle number operator is naturally expressed in normal order and, in the pertinent scenario, the associated representation of the phase space distribution is the Glauber–Sudarshan  $P$  one [3]. In addition to  $W$  and  $P$ , one may find many other quasi-probability distributions emerging in alternative representations of the phase space distribution [5]. A quite popular representation is the Husimi  $Q$  one [6–9], used when operators are in anti-normal order. We emphasize that we work here with classical analogs of  $W$ ,  $P$ , and  $Q$ . As stated, we will specialize things to the three  $f$ -functions associated to a Harmonic Oscillator (HO) of angular frequency  $\omega$ . In such a scenario the three (classical analog) functions –that we call for convenience  $f_P$ ,  $f_Q$ , and  $f_W$ – are just Gaussians. The pertinent treatment becomes wholly analytical.

## A. Our goal

In this paper we wish to apply *semiclassical information theory tools* associated to these analog  $P$ ,  $Q$ , and  $W$  representations (for quadratic Hamiltonians) *in order to describe the associated thermal semiclassical features*. The idea is to gain physical insight from the application of different information-quantifiers to classical analogs of quasi-probability distributions. It will be seen that useful insights are in this way gained. WE will discover that out of the three functions, only  $f_Q$  and  $f_W$  are sensible analogs, while  $f_P$  exhibits problems if the temperature is low enough.

We insist. In his paper we will regard quasi-probabilities as semi-classical distributions in phase space –analogs of the quantum quasi-probabilistic distributions–, and try to ascertain what physical features are they able to describe at such semi-classical level. One has [15, 16]:

$$f_P = \gamma_P e^{-\gamma_P |\alpha|^2}, \quad \gamma_P = e^{\beta \hbar \omega} - 1 \quad (P - \text{function}), \quad (2)$$

$$f_Q = \gamma_Q e^{-\gamma_Q |\alpha|^2}, \quad \gamma_Q = 1 - e^{-\beta \hbar \omega} \quad (Q - \text{function}), \quad (3)$$

$$f_W = \gamma_W e^{-\gamma_W |\alpha|^2}, \quad \gamma_W = 2 \tanh(\beta \hbar \omega / 2) \quad (W - \text{function}), \quad (4)$$

with  $\beta = 1/k_B T$ ,  $k_B$  the Boltzmann constant, and  $T$  the temperature. They will be used as semiclassical statistical weight functions. Since ours is not a quantum approach, the ordering of the HO-creation and destruction operators  $\hat{a}$  and  $\hat{a}^\dagger$  plays no role whatsoever.

## B. Historic considerations and organization

The thermodynamics properties associated to coherent states have been the subject of much interest. See, for instance, Refs. [10] and [11]). Notice that the HO is a really relevant system that yields useful insights of wide impact. Indeed, the HO constitutes much more than a mere example. It is of special relevance for bosonic or fermionic atoms contained in magnetic traps [12–14], as well as for system that exhibit an equidistant level spacing in the vicinity of the ground state, like nuclei or Luttinger liquids.

For thermal states, the gaussian HO-quantum phase spaces distributions are known in the literature for applications in quantum optics.

This paper is organized as follows: section II refers to different information quantifiers in a phase space representation for Gaussian distributions. In Section we calculate the classical analog-Fano factor. Features of the fluctuations are analyzed in Section IV. Also, we discuss the notion of linear entropy. Finally, some conclusions are drawn in Section V.

## II. SEMI-CLASSICAL INFORMATION QUANTIFIERS

Consider a general normalized gaussian distribution in phase space

$$f(\alpha) = \gamma e^{-\gamma |\alpha|^2}, \quad (5)$$

whose normalized variance is  $1/\gamma$  and  $\gamma$  taking values  $\gamma_P$ ,  $\gamma_Q$  and  $\gamma_W$ . We discuss next, in these terms, some important information theory quantifiers.

### A. Shift-invariant Fisher's information measure

The information quantifier Fisher's information measure, specialized for families of shift-invariant distributions, that do not change shape under translations, is [17, 18]

$$I = \int dx f(x) \left( \frac{\partial \ln f(x)}{\partial x} \right)^2, \quad (6)$$

and, in phase space, adopts the appearance [19]

$$I = \frac{1}{4} \int \frac{d^2 \alpha}{\pi} f(\alpha) \left( \frac{\partial \ln f(\alpha)}{\partial |\alpha|} \right)^2, \quad (7)$$

such that considering  $f(\alpha)$  given by Eq. (5) we get  $I = \gamma$ , whose specific values are  $\gamma_P$ ,  $\gamma_Q$ ,  $\gamma_W$  for the three functions  $f_P$ ,  $f_Q$ , and  $f_W$ . The behavior of these quantities are displayed in Fig. 1. The solid line is the case P, the dashed one the Wigner one, and the dotted curve is assigned to the Husimi case. Now, it is known that in the present scenario

the maximum attainable value for  $I$  equals 2 [19]. The  $P$ -result violates this restriction at low temperatures, more precisely at

$$T < T_{crit} = (\hbar\omega/k_B)/\ln 3 \approx 0.91023\hbar\omega/k_B, \quad (8)$$

with  $T$  being expressed in  $(\hbar\omega/k_B)$ -units.

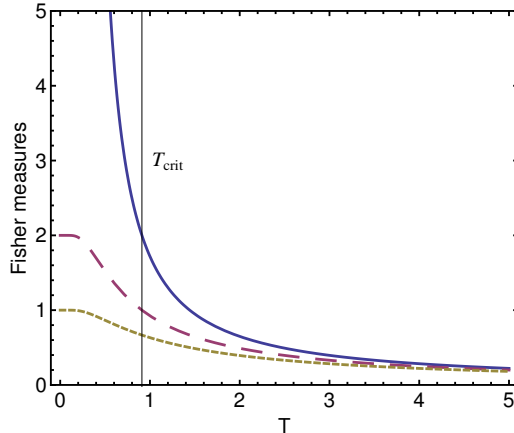


FIG. 1: Fisher measure versus temperature  $T$ , expressed in  $(\hbar\omega/k_B)$ -units. The solid line is the case  $P$ , the dashed one the Wigner one, and the dotted curve is assigned to the Husimi instance. The vertical line represents the critical temperature  $T_{crit}$ .

### B. Logarithmic entropy $S$

The logarithmic Boltzmann's information measure for the the probability distribution (5) is

$$S = - \int \frac{d^2\alpha}{\pi} f(\alpha) \ln f(\alpha) = 1 - \ln \gamma, \quad (9)$$

so that it acquires the particular values

$$S_P = 1 - \ln(e^{\beta\hbar\omega} - 1), \quad (10)$$

$$S_Q = 1 - \ln(1 - e^{-\beta\hbar\omega}), \quad (11)$$

$$S_W = 1 - \ln(2 \tanh(\beta\hbar\omega/2)), \quad (12)$$

for, respectively, the distributions  $f_P$ ,  $f_Q$ , and  $f_W$ . These entropies are plotted in Fig. 2. Notice that  $S_P < 0$  for  $T < \hbar\omega/(k_B \ln(1 + e)) \approx 0.76(\hbar\omega/k_B) < T_{crit}$ . Negative classical entropies are well-known. One can cite, as an example, Ref. [20].

Table I lists a set of critical temperatures  $T_{crit}$  for typical electromagnetic (EM) waves.

We see from Table I that serious anomalies are detected for the  $P$ -distribution in the case of radio waves of high frequency.  $P$  becomes negative, which is absurd, for rather high temperatures, where one expects classical physics to reign. Accordingly, one concludes that quasi-probabilities do not exhibit a sensible classical limit in the  $P$ -case, contrary to what happens in both the  $W$  and  $Q$  ones.

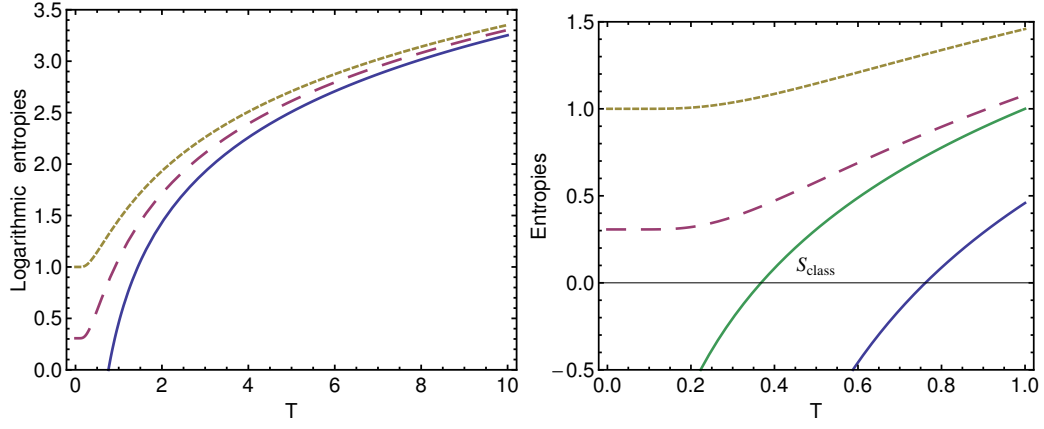


FIG. 2: Left: logarithmic entropies  $S_P$ ,  $S_Q$ , and  $S_W$ , as a function of the temperature  $T$  in  $(\hbar\omega/k_B)$ -units. The solid line is the case  $P$ , the dashed one the Wigner one, and the dotted curve is assigned to the Husimi instance. Right: zoom of the logarithmic entropies as a function of temperature  $T$  in  $(\hbar\omega/k_B)$ -units. Negative values of  $S_P$  occur below  $T = \hbar\omega/(k_B \ln(1 + e))$ , the critical temperature  $< T_{crit}$ . Remaining details are similar to those of left figure. We have added the classical entropy of the harmonic oscillator  $S_{class} = 1 - \ln(\beta\hbar\omega)$ .

	frequency ( $\nu$ )	Critical temperatures ( $^{\circ}K$ )
Extremely low frequency ELF	3 – 30Hz	$1.4397 \cdot 10^{-10} - 1.4397 \cdot 10^{-9}$
Super low frequency SLF	30 – 300Hz	$1.4397 \cdot 10^{-9} - 1.4397 \cdot 10^{-8}$
Ultra low frequency ULF	300 – 3000Hz	$1.4397 \cdot 10^{-8} - 1.4397 \cdot 10^{-7}$
Very low frequency VLF	3 – 30kHz	$1.4397 \cdot 10^{-7} - 1.4397 \cdot 10^{-6}$
Low frequency LF	30 – 300kHz	$1.4397 \cdot 10^{-6} - 1.4397 \cdot 10^{-5}$
Medium frequency MF	300kHz – 3MHz	$1.4397 \cdot 10^{-5} - 1.4397 \cdot 10^{-4}$
High frequency HF	3 – 30MHz	$1.4397 \cdot 10^{-4} - 1.4397 \cdot 10^{-3}$
Very high frequency VHF	30 – 300MHz	$1.4397 \cdot 10^{-3} - 1.4397 \cdot 10^{-2}$
Ultra high frequency UHF	300MHz – 3GHz	$1.4397 \cdot 10^{-2} - 1.4397 \cdot 10^{-1}$
Super high frequency SHF	3 – 30GHz	$1.4397 \cdot 10^{-1} - 1.4397$
Extremely high frequency EHF	30 – 300GHz	$1.4397 - 14.397$
Tremendously high frequency THF	300GHz – 3000GHz	$14.397 - 143.97$

TABLE I: Critical temperatures  $T_{crit}$  for typical radio waves, with  $h/k_B = 4.799 \cdot 10^{-11}$  Kelvin per second.

### C. Statistical complexity

The statistical complexity  $C$ , according to Lopez-Ruiz, Mancini, and Calvet [22], is a suitable product of two quantifiers, such that  $C$  becomes minimal at the extreme situations of perfect order or total randomness. Instead of using the prescription of [22], but without violating its spirit, we will take one of these two quantifiers to be Fisher's measure and the other an entropic form, since it is well known that the two behave in opposite manner [23]. Thus:

$$C = SI = \gamma(1 - \ln \gamma), \quad (13)$$

that vanishes for perfect order or total randomness. For each particular case, we explicitly have

$$C_P = (e^{\beta\hbar\omega} - 1) [1 - \ln(e^{\beta\hbar\omega} - 1)], \quad (14)$$

$$C_Q = (1 - e^{-\beta\hbar\omega}) [1 - \ln(1 - e^{-\beta\hbar\omega})], \quad (15)$$

$$C_W = 2 \tanh(\beta\hbar\omega/2) [1 - \ln(2 \tanh(\beta\hbar\omega/2))], \quad (16)$$

for, respectively, the distributions  $f_P$ ,  $f_Q$ , and  $f_W$ . The maximum of the statistical complexity occurs when  $\gamma = 1$  and, the associated temperature values are

$$\left\{ \begin{array}{ll} e^{\beta\hbar\omega} - 1 = 1 \Rightarrow T = \hbar\omega/k_B \ln 2 > T_{crit} & \text{for the } f_P\text{-function,} \\ 1 - e^{-\beta\hbar\omega} = 1 \Rightarrow T = 0 & \text{for the } f_Q\text{-function,} \\ 2 \tanh(\beta\hbar\omega/2) = 1 \Rightarrow T = \hbar\omega/2k_B \arctan(1/2) & \text{for the } f_W\text{-function.} \end{array} \right.$$

The statistical complexity  $C$  is plotted in Fig. 3.

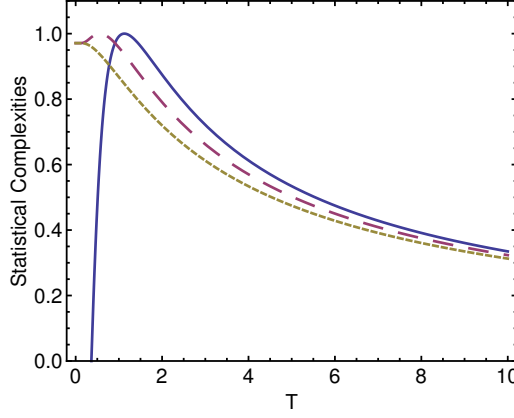


FIG. 3: Complexities  $C_P$ ,  $C_Q$ , and  $C_W$  versus the temperature  $T$  in  $(\hbar\omega/k_B)$ -units. The solid line is the case  $P$ , the dashed one the Wigner one, and the dotted curve is assigned to the Husimi instance.

#### D. Linear entropy

Another interesting information quantifier is that of the Manfredi-Feix entropy [24], derived from the phase space Tsallis ( $q = 2$ ) entropy [25]. In quantum information this form is referred to as the linear entropy [26]. It reads

$$S_l = 1 - \int \frac{d^2\alpha}{\pi} f^2(\alpha) = 1 - \mathcal{J}, \quad (17)$$

$$\mathcal{J} = \int \frac{d^2\alpha}{\pi} f^2(\alpha) = \frac{\gamma}{2}. \quad (18)$$

Accordingly, we have

$$S_l = 1 - \frac{\gamma}{2}; \quad 0 \leq S_l \leq 1. \quad (19)$$

This is semi-classical result, valid for small  $\gamma$ . In particular,

$$S_{l,P} = 1 - \frac{\gamma_P}{2} = 1 - \frac{e^{\beta\hbar\omega} - 1}{2}, \quad (20)$$

$$S_{l,Q} = 1 - \frac{\gamma_Q}{2} = 1 - \frac{1 - e^{-\beta\hbar\omega}}{2}, \quad (21)$$

$$S_{l,W} = 1 - \frac{\gamma_W}{2} = 1 - \frac{2 \tanh(\beta\hbar\omega/2)}{2}. \quad (22)$$

Note that in the  $P$ -instance the linear entropy becomes negative, once again, for  $T < T_{crit}$ . Contrary to what happens for the logarithmic entropy, the linear one can vanish in the  $W$  representation.

The ensuing statistical complexity that uses  $S_l$  becomes

$$C_l = S_l I = I \left(1 - \frac{I}{2}\right) = \gamma \left(1 - \frac{\gamma}{2}\right), \quad (23)$$

vanishing both for  $\gamma = 0$  and for  $\gamma = 2$ , the extreme values of the  $\gamma$ -physical range (we showed above that  $\gamma$  cannot exceed 2 without violating uncertainty restrictions).

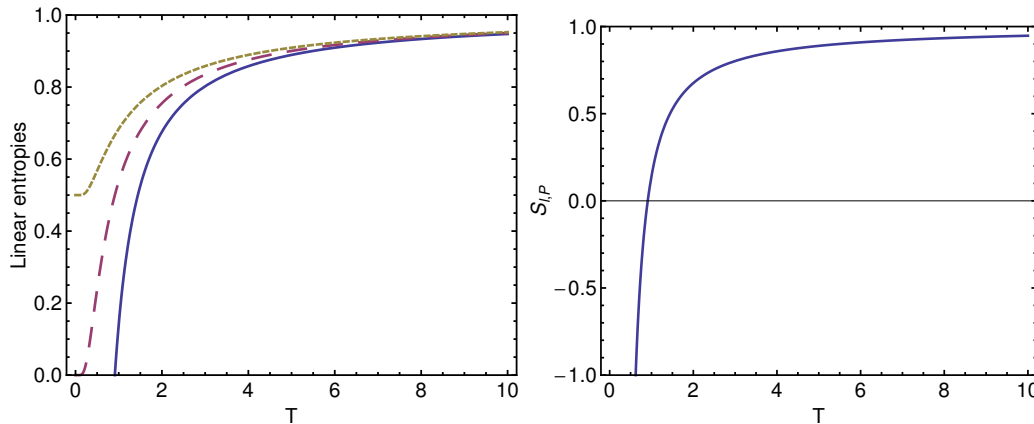


FIG. 4: Left: Linear entropies  $S_{l,P}$ ,  $S_{l,Q}$ , and  $S_{l,W}$  versus the temperature  $T$  in  $(\hbar\omega/k_B)$ -units. The solid line is the case  $P$ , the dashed one the Wigner one, and the dotted curve is assigned to the Husimi instance. Right:  $S_{l,P}$  as a function of the temperature  $T$  in  $(\hbar\omega/k_B)$ -units. We note the linear entropy  $S_{l,P}$  is negative below the  $T$ -value  $t_{crit} = T_{crit}k_B/\hbar\omega = \ln 3$ .

### III. FANO FACTOR'S CLASSICAL ANALOG

In general, the Fano factor is the coefficient of dispersion of the probability distribution  $p(y)$ , which is defined as [21]

$$\mathcal{F} = \frac{\Delta y^2}{\langle y \rangle}, \quad (24)$$

where  $\Delta y^2 = \langle y^2 \rangle - \langle y \rangle^2$  is the variance and  $\langle x \rangle$  is the mean of a random process  $y$ .

If  $p(y)$  is a Poisson distribution then one sees that the pertinent Fano factor becomes unity ( $\mathcal{F} = 1$ ) [15, 27]. We remind the reader of two situations:

1. for  $\mathcal{F} < 1$ , sub-Poissonian processes occur, while
2. for  $\mathcal{F} > 1$ , corresponds to a super-Poissonian process.

For our gaussian distribution (5), if one sets now  $y = |\alpha|^2$  one has the classical Fano-analog

$$\mathcal{F} = \frac{\langle |\alpha|^4 \rangle_f - \langle |\alpha|^2 \rangle_f^2}{\langle |\alpha|^2 \rangle_f}, \quad (25)$$

where the expectation value of the function  $\mathcal{A}(\alpha)$  is calculated as

$$\langle \mathcal{A} \rangle_f = \int \frac{d^2\alpha}{\pi} f(\alpha) \mathcal{A}(\alpha), \quad (26)$$

indicating that  $f(\alpha)$  is the statistical weight function. Thus, after computing the mean values involved in Eq. (25) by taking into account the definition (26), the Fano factor becomes

$$\mathcal{F} = \frac{1}{\gamma} = \frac{1}{I}, \quad (27)$$

which, for a Gaussian distribution, links the Fano factor to the distribution's width and to the Fisher's measure  $I$ . We are speaking of processes that are of a quantum nature and can not take place in a classical environment. Thus, with reference to the critical temperature defined in Eq. (8) we have to deal with

$$\mathcal{F}_P = \frac{1}{e^{\beta\hbar\omega} - 1} \quad (= 1 \text{ at } T = \frac{\hbar\omega}{k_B \ln 2} > T_{crit}) \quad \text{for the } f_P \text{ function,} \quad (28)$$

$$\mathcal{F}_Q = \frac{1}{1 - e^{-\beta\hbar\omega}} \quad (= 1 \text{ at } T = 0) \quad \text{for the } f_Q \text{ function,} \quad (29)$$

$$\mathcal{F}_W = \frac{1}{2 \tanh(\beta\hbar\omega/2)} \quad (= 1 \text{ at } T = T_{crit}) \quad \text{for the } f_W \text{ function.} \quad (30)$$

The  $f_Q$ -case reaches the *super to sub-Poissonian transition* only at  $T = 0$ , while the other two cases reach it at finite temperatures.

#### IV. FLUCTUATIONS

We start this section considering the classical Hamiltonian of the harmonic oscillator that reads

$$\mathcal{H}(x, p) = \hbar\omega|\alpha|^2, \quad (31)$$

where  $x$  and  $p$  are phase space variables,  $|\alpha|^2 = x^2/4\sigma_x^2 + p^2/\sigma_p^2$ , and  $\sigma_x^2 = \hbar/2m\omega$  and  $\sigma_p^2 = \hbar m\omega/2$  [28].

Using the definition of the mean value (26), from (31) we immediately find [29]

$$\langle x^2/2\sigma_x^2 \rangle_f = \langle p^2/2\sigma_p^2 \rangle_f = \langle |\alpha|^2 \rangle_f, \quad (32)$$

with

$$\langle |\alpha|^2 \rangle_f = \gamma \int \frac{d^2\alpha}{\pi} e^{-\gamma|\alpha|^2} |\alpha|^2 = \frac{1}{\gamma}, \quad (33)$$

where  $\langle x \rangle_f = \langle p \rangle_f = \langle \alpha \rangle_f = 0$ , while  $\gamma$  takes the respective values  $\gamma_P$ ,  $\gamma_Q$ , and  $\gamma_W$ . The concomitant variances are  $\Delta x^2 = \langle x^2 \rangle_f - \langle x \rangle_f^2 = 2\sigma_p^2/\gamma$ , and  $\Delta p^2 = \langle p^2 \rangle_f - \langle p \rangle_f^2 = 2\sigma_x^2/\gamma$ . Hence, for our general gaussian distribution one easily establishes that

$$\mathcal{U} = \Delta x \Delta p = \frac{\hbar}{\gamma}, \quad (34)$$

which shows that  $\gamma$  should be constrained by the restriction

$$\gamma \leq 2, \quad (35)$$

if one wishes the inequality

$$\Delta x \Delta p \geq \hbar/2, \quad (36)$$

to hold.

Specializing (34) for our three quasi-probability distributions yields

$$\Delta x \Delta p = \frac{\hbar}{e^{\beta \hbar \omega} - 1}, \quad \text{for } f_P \text{ function,} \quad (37)$$

$$\Delta x \Delta p = \frac{\hbar}{1 - e^{-\beta \hbar \omega}}, \quad \text{for } f_Q \text{ function,} \quad (38)$$

$$\Delta x \Delta p = \frac{\hbar}{2 \tanh(\beta \hbar \omega / 2)}, \quad \text{for } f_W \text{ function.} \quad (39)$$

The restriction (36) applied to the  $P$ -result entails that it holds if

$$T \geq \frac{\hbar \omega}{\ln 3 k_B} = T_{crit} \approx 0.91023 \frac{\hbar \omega}{k_B}. \quad (40)$$

Thus, the distribution  $f_P$  seems again to become un-physical at temperatures lower than  $T_{crit}$ , for which (36) is violated. From (34) we have  $\gamma = \hbar/\mathcal{U}$ . Accordingly, if we insert this into (9), the logarithmic entropy  $S$  can be recast in  $\mathcal{U}$ -terms via the relation

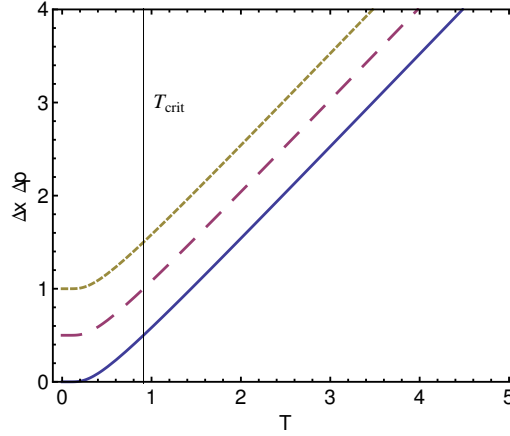


FIG. 5: Fluctuations vs. the temperature  $T$  in  $(\hbar\omega/k_B)$ -units. The solid line is the case  $P$ , the dashed one the Wigner one, and the dotted line is assigned to the Husimi instance.

$$S = 1 - \ln \left( \frac{\hbar}{\Delta x \Delta p} \right), \quad (41)$$

(also demonstrated in Ref. [30] to hold for the Wehrl entropy) that vanishes for

$$\Delta x \Delta p = \frac{\hbar}{e}. \quad (42)$$

In the  $P$ -instance this happens at

$$T = 0.71463 \frac{\hbar \omega}{k_B}. \quad (43)$$

At this temperature the Heisenberg's-like condition (36) is violated. The  $W$  and  $Q$  distributions do not allow for such a circumstance. Actually, in the Wigner case, which is exact, the minimum  $S$ -value is attained at  $\beta = \infty$ , where

$$S_{min} = 1 - \ln 2 \approx 0.306. \quad (44)$$

The uncertainty restriction (36) seems to impede the phase-space entropy to vanish, a sort of quasi-quantum effect. It is clear then that, in phase space, the logarithmic entropy, by itself, is an uncertainty indicator, in agreement with the work, in *other scenarios*, of several authors (see, for instance, [31] and references therein).



Define now the participation ratio's analog as [32, 33]

$$m = \frac{1}{\mathcal{J}} = \frac{2}{\gamma}, \quad (45)$$

where  $\mathcal{J}$  is given by (18). This is an important quantity that measures the number of pure states entering the mixture determined by our general gaussian probability distribution of amplitude  $\gamma$  [32, 33]. We again encounter troubles with the  $P$ -distribution in this respect. It is immediately realized by looking at Fig. 6 that, for fulfilling the obvious condition  $m \geq 1$ , one needs a temperature  $T \geq T_{crit}$ .

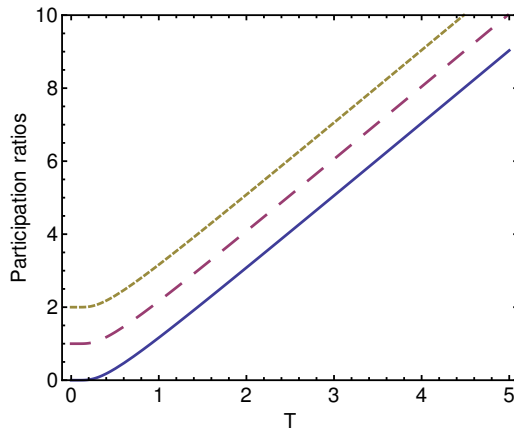


FIG. 6: Participation ratio  $m$  versus temperature  $T$  in  $(\hbar\omega/k_B)$ -units. The solid line is the case  $P$ , the dashed one the Wigner one, and the dotted curve is assigned to the Husimi instance.

## V. CONCLUSIONS

We have investigated here the thermal statistics of quasi-probabilities-analogs  $f(\alpha)$  in phase space for the important case of quadratic Hamiltonians, focusing attention on the three more important instances, i.e., those of Wigner,  $P$ -, and Husimi distributions.

- We emphasized the fact that for all of them the semiclassical entropy is a function only of the fluctuation-product  $\Delta x \Delta p$ . This fact allows one to ascertain that the analog  $P$ -distribution seems to become un-physical at low enough temperatures, smaller than a critical value  $T_{crit}$ , because, in such an instance:
  1. it would violate a Heisenberg's-like principle in such a case. The behavior of other information quantifiers reconfirms such an assertion, i.e.,
  2. Fisher's measure exceeds its permissible maximum value  $I = 2$ ,
  3. the participation ratio becomes  $< 1$ , which is impossible.
- It is also clear then that semiclassical entropy, by itself, in phase space, looks like a kind of “uncertainty” indicator.
- We have determined the temperatures for which the statistical complexity becomes maximal, as a signature of the well-known transition between classical and nonclassical states of light whose signature is the transition from super-Poissonian to sub-Poissonian distributions [34].

We have seen that the  $P$ -distribution in the case of radio waves of high frequency becomes negative, which is absurd, as the pertinent temperatures are rather high and thus one expects classical physics to reign. Accordingly, one concludes that quasi-probabilities do not exhibit a sensible classical limit in the  $P$ -case, contrary to what happens in both the  $W$  and  $Q$  ones.

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